Pareto optimization for resonances

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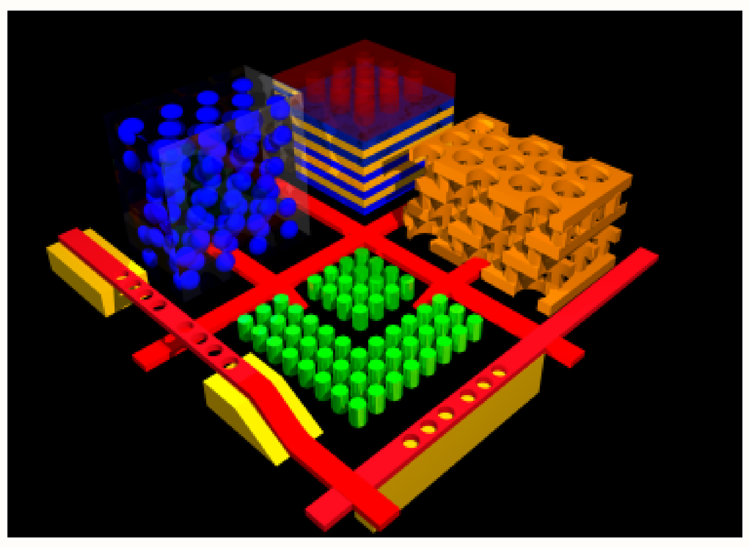
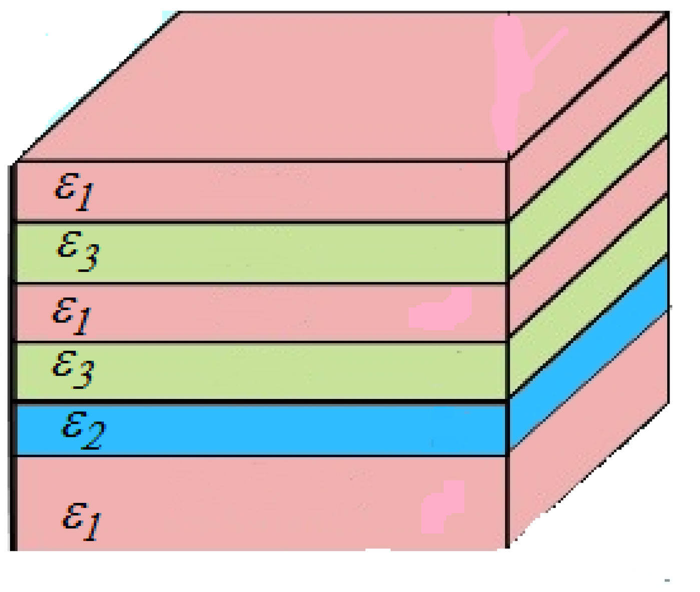
Resonances in 1-D photonic crystals (PC).



from the cover of "Photonic Crystals: Molding the Flow of Light" by Joannopoulos, Johnson, Winn, Meade '08



1-D photonic crystal

PC are periodic (or close to periodic) metallo-dielectric nanostructures affecting EM waves propagation governed by

$$\partial_t \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon_0 \varepsilon(x)} \operatorname{rot} \\ -\frac{1}{\mu_0} \operatorname{rot} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix},$$

$$\varepsilon(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \notin \mathcal{D} \\ \text{nonhomogeneous structure}, & \mathbf{x} \in \mathcal{D} \end{cases}.$$

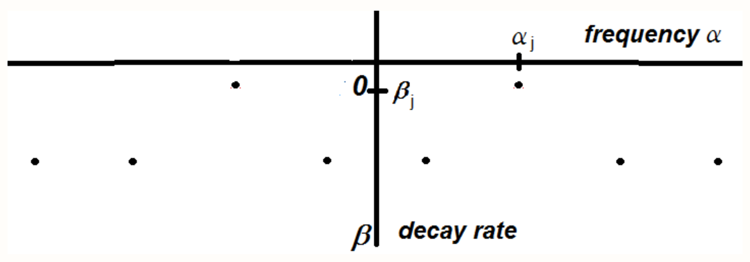
Solutions $e^{-i\omega t} \begin{pmatrix} \mathbf{E}(\mathbf{x}) \\ \mathbf{H}(\mathbf{x}) \end{pmatrix}$ with outgoing $\begin{pmatrix} \mathbf{E}(\mathbf{x}) \\ \mathbf{H}(\mathbf{x}) \end{pmatrix}$ correspond to resonances ω.

In 1-D case, resonances ω are eigenvalues of

$$y''(x) = -\omega^2 B(x) y(x), \quad 0 < x < \ell, \qquad (1)$$

equipped with radiation boundary conditions.

Let $\Sigma(B)$ be the set of resonances associated with a structure $B(x)$.

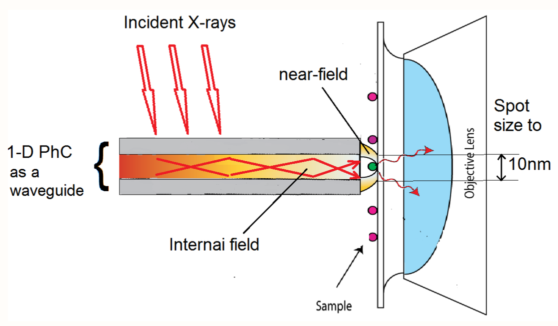


Motivation for optimization

Normally passing waves create the electric field $\mathbf{E} = (0, E_2, 0)$:

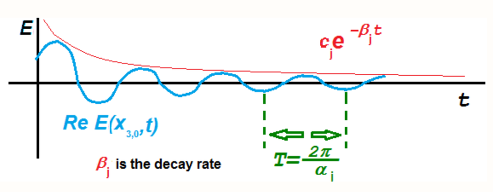
$$E_2(x_3, t) = \sum_{k=-\infty}^{+\infty} c_k \psi_k(x_3) e^{-i\omega_k t},$$

$\omega_k = \alpha_k - i\beta_k \in \mathbb{C}_-$ are resonances,



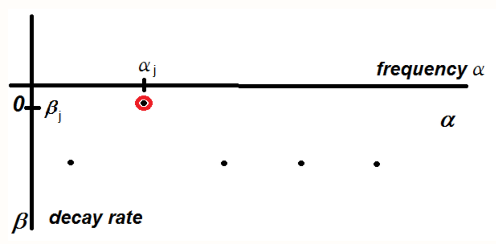
$\beta_k = -\operatorname{Im} \omega_k > 0$ is the decay rate,

$\alpha_k = \operatorname{Re} \omega_k$ is the angular frequency.



Assume that for a certain j, $\beta_j \ll \inf_{k \neq j} \beta_k$,

then $E_2 \approx c_j \psi_j(x_3) e^{-i\omega_j t}$ for large t (under certain additional assumptions).

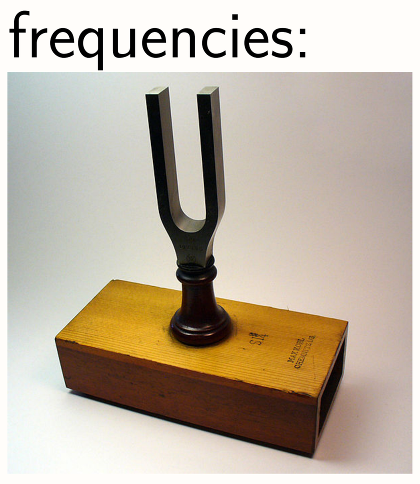


The field strength is essentially determined by $\beta_j = -\operatorname{Im} \omega_j$.

Problem. **Minimization of the decay rate $\beta = |\operatorname{Im} \omega|$** for a given range of frequencies α.

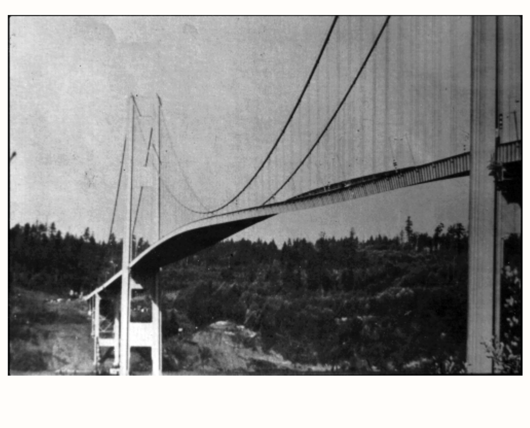
Related optimization problems

Specially designed resonators for particular frequencies:



Turning fork, by Max Kohl, Chemnitz, Germany. CWRU Physics Dept.

Unwanted mechanical resonances:



The first Tacoma Narrows Bridge on the day of its collapse (http://en.wikipedia.org/wiki/Tacoma_Narrows_Bridge_1940)

Rigorous approach to optimization

Consider two types of families $\mathbb{A}$ of admissible structures. The resonator structure is represented by either a function $B(x)$, or a measure dM.

- Side constraints, $0 \le b_1 \le B(x) \le b_2$.

$$\mathbb{A}_\infty = \{ B(x) \in L^\infty(0, \ell) \ : \ b_1 \le B(x) \le b_2 \quad \text{a.e.} \}.$$

- Total mass constraints for a Borel measure dM:

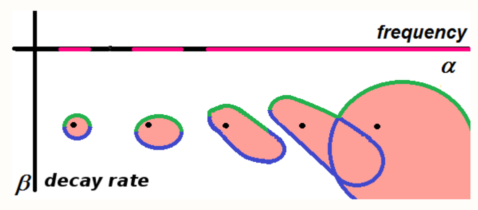
$$\mathbb{A}_\mathbb{M} = \{ \text{nonnegative } dM \ : \ \int_{0-}^{\ell+} dM \le m \}, \quad m > 0.$$

In the latter case, equation (1) has to be generalized to

$$\frac{d^2 y'}{dM dx} = -\omega^2 y(x), \quad 0 < x < \ell.$$

The set of admissible resonances. $\Sigma[\mathbb{A}] := \cup_{B \in \mathbb{A}} \Sigma(B)$

For small dielectric contrast $b_2 - b_1 \ll b_2$:



A frequency α is admissible if $\alpha = \operatorname{Re} \omega$ for certain $\omega \in \Sigma[\mathbb{A}]$.

Minimal decay for admissible α

$$\beta_{\min}(\alpha) = \min_{\substack{\operatorname{Re} \omega = \alpha \\ \omega \in \Sigma(\mathbb{A})}} |\operatorname{Im} \omega|$$

Green curves above are resonances of minimal decay. They always exist for optimization problems over $\mathbb{A}_\infty$ and $\mathbb{A}_\mathbb{M}$.

Main results: structural theorems and reduction to 4-D

Theorem (IK, Asymptotic Analysis '13)

Resonators of minimal decay for a frequency α under the side constraints are extremal PC.

That is, optimal $B(x)$ over $\mathbb{A}_\infty$ are piecewise constant and take only b_1 and b_2.

Positions of optimal switch points between media b_1 and b_2 are described by the nonlinear equation

$$y'' = -\omega^2 y \left[b_1 + (b_2 - b_1) \chi_{\mathbb{C}_+}(y^2) \right],$$

where $\chi_{\mathbb{C}_+}(\zeta) := 1$ if $\operatorname{Im} \zeta > 0$, and $\chi_{\mathbb{C}_+}(\zeta) = 0$ if $\operatorname{Im} \zeta \le 0$.

Theorem (IK, J. Differential Equations '14)

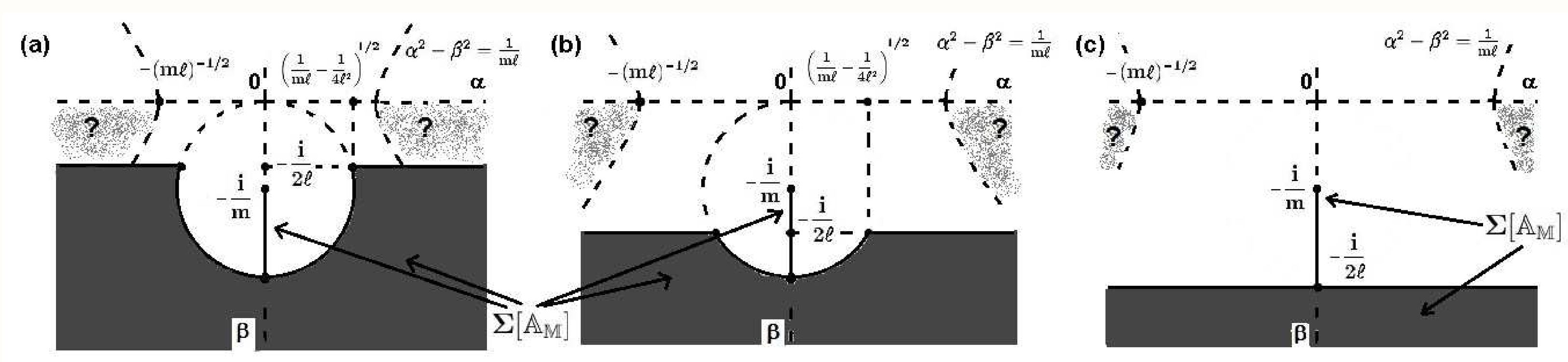
Let dM_0 be of minimal decay over $\mathbb{A}_\mathbb{M}$. Then dM_0 consists of a finite number of point masses.

Reduce the problems to optimization over 3 and 4 real parameters, resp.

Calculation of optimal ω over $\mathbb{A}_\mathbb{M}$

Additionally, strong restrictions on optimal masses m_j and their positions a_j are obtained.

For low frequencies, optimizers and $\Sigma[\mathbb{A}_\mathbb{M}]$ are explicitly found:



(a) $\ell < m < 2\ell$, **(b)** $2\ell < m < 4\ell$, **(c)** $4\ell \le m$.
The locations of the circle $|\omega + i/m| = 1/m$, the line $\operatorname{Im} \omega = -\frac{1}{2\ell}$, and the hyperbola $\operatorname{Re} \omega^2 = \frac{1}{m\ell}$ are given schematically.