CYCLES OF LINEAR AND SEMILINEAR MAPPINGS

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Semilinear mappings

A mapping $\mathcal{A}$ from a complex vector space $\mathbf{U}$ to a complex vector space $\mathbf{V}$ is semilinear if

$$\mathcal{A}(u+u') = \mathcal{A}u + \mathcal{A}u', \qquad \mathcal{A}(\alpha u) = \bar{\alpha}\mathcal{A}u$$

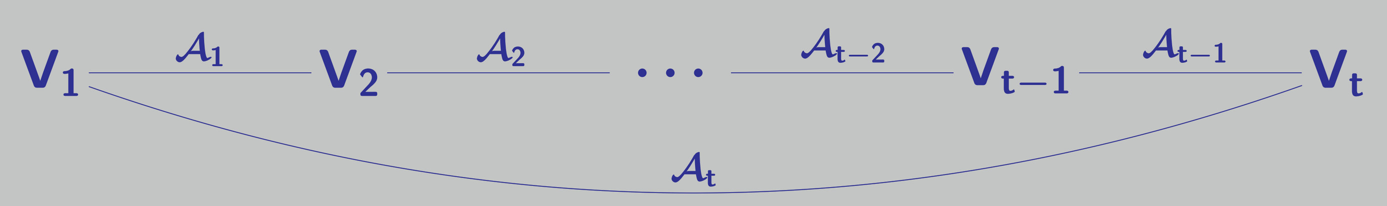
for all $u, u' \in \mathbf{U}$ and $\alpha \in \mathbb{C}$.

We write

- $\mathcal{A} : \mathbf{U} \longrightarrow \mathbf{V}$ if $\mathcal{A}$ is a linear mapping, and
- $\mathcal{A} : \mathbf{U} \dashrightarrow \mathbf{V}$ if $\mathcal{A}$ is a semilinear mapping.

Abstract

We give a canonical form of matrices of a cycle of linear and semilinear mappings

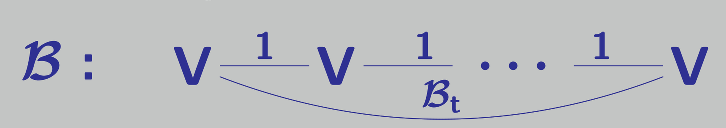


in which each line is

- a full arrow $\longrightarrow$, $\longleftarrow$, or
- a dashed arrow $\dashrightarrow$, $\dashleftarrow$.

All $\mathcal{A}_i$ are nonsingular

Then $\mathcal{A}$ is isomorphic to a cycle



whose classification reduces to the classification of

- linear operators $\mathbf{V} \circlearrowleft \mathcal{B}_t$ if the number of dashed arrows in $\mathcal{A}$ is even;
- semilinear operators $\mathbf{V} \circlearrowleft \mathcal{B}_t$ (dashed) if the number of dashed arrows in $\mathcal{A}$ is odd.

The classification of linear operators is given by Jordan's theorem.

The classification of semilinear operators

The matrix of a semilinear operator is transformed by consimilarity transformations $\mathbf{A} \mapsto \bar{\mathbf{S}}^{-1}\mathbf{A}\mathbf{S}$. Its canonical form is given in

- Y.P. Hong, R.A. Horn, *A canonical form for matrices under consimilarity*, Linear Algebra Appl. 102 (1988) 143–168.
- R.A. Horn, C.R. Johnson, *Matrix Analysis, 2nd ed.*, Cambridge University Press, New York, 2012.

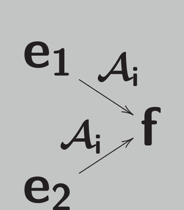
Each square complex matrix is consimilar to a direct sum, uniquely determined up to permutation of direct summands, of matrices of the following types:

- $J_k(\lambda)$, *in which* $\lambda \geq 0$, *and*
- $\begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix}$, *in which* $\mu \notin \mathbb{R}$ *or* $\mu < 0$.

There is a singular $\mathcal{A}_i$

Lemma 3
There are bases of $\mathbf{V}_1, \dots, \mathbf{V}_t$ *in which the matrix* $\mathbf{A}_i$ *of each* $\mathcal{A}_i$ *possesses the property: all entries are only 0's and 1's with at most one 1 in each row and each column.*

- Each $\mathcal{A}_i$ maps a basis vector to a basis vector or $\mathbf{0}$.
- Each $\mathcal{A}_i$ cannot map distinct basis vectors to the same basis vector:



— impossible

(i.e., the arrows must be parallel)

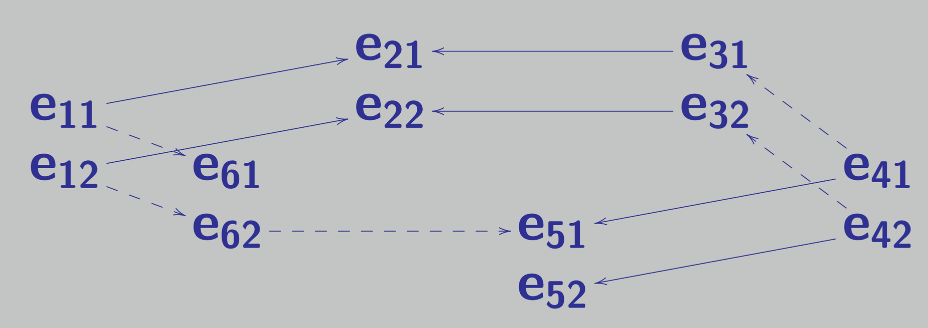
Construct the directed graph,

- its vertices are basis vectors of $\mathbf{V}_1, \dots, \mathbf{V}_t$ from Lemma 3,
- there is an arrow from u to v if and only if $\exists \mathcal{A}_i : u \mapsto v$.

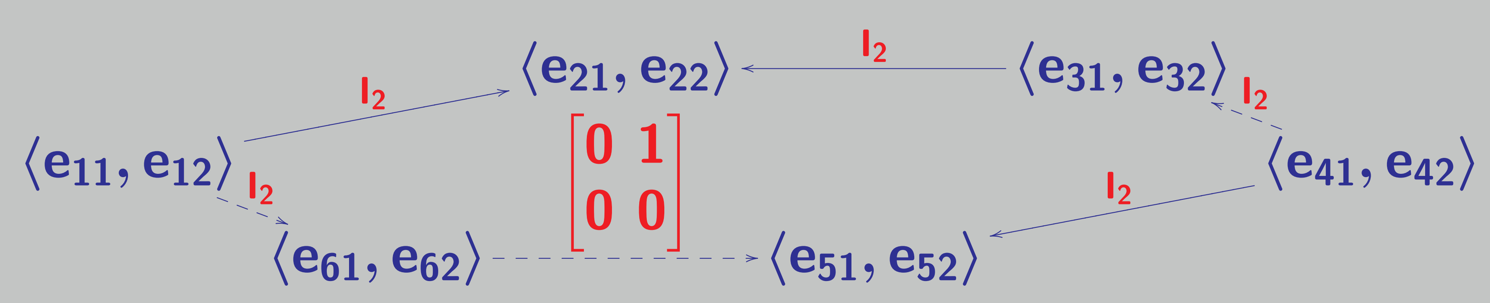
By Lemma 3, the graph is a disjoint union of chains.
Since $\mathcal{A}$ is indecomposable, the graph is connected, and so it is a chain.

Type I: The chain stops exactly before the starting point

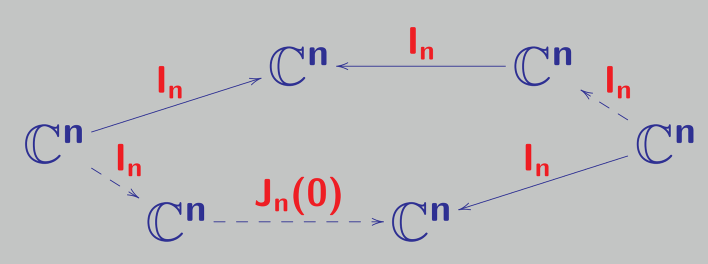
An example:



Then



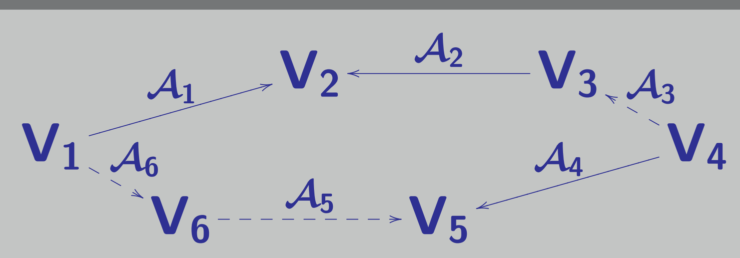
The general case:



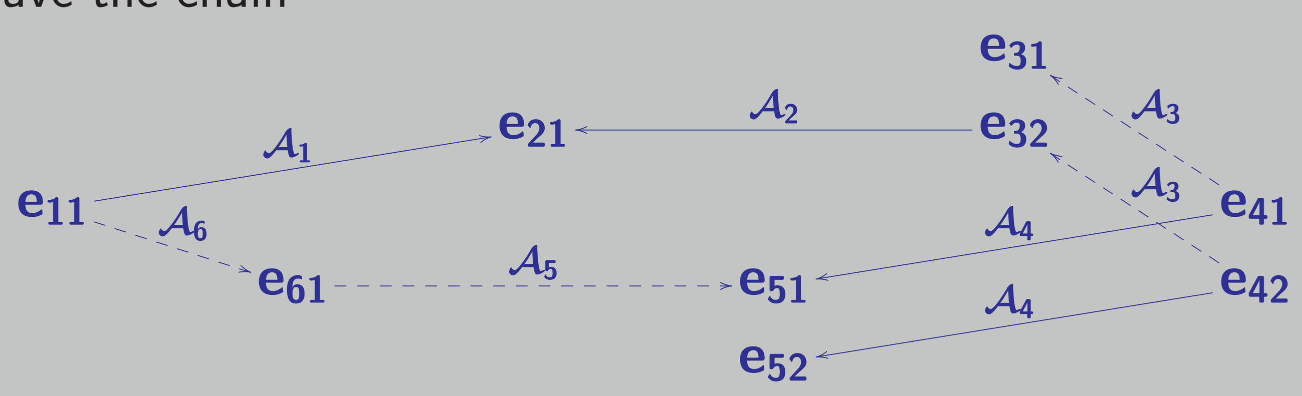
The singular Jordan can be over any arrow.

Type II: The chain does not stop exactly before the starting point

An example: A cycle



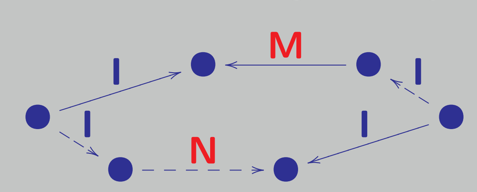
may have the chain



Its mappings $\mathcal{A}_1, \dots, \mathcal{A}_6$ are given by the matrices

$$A_1 = A_6 = [1], \quad A_3 = A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The general case: Two arrows are assigned by $\mathbf{M}$ and $\mathbf{N}$; the others by $\mathbf{I}$:



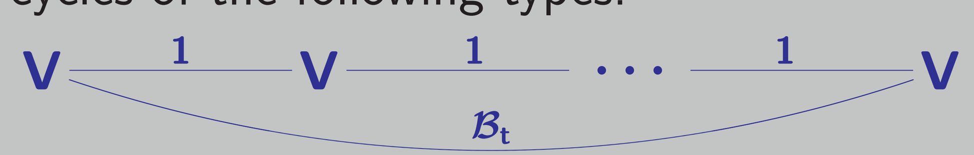
$$(M, N) = \begin{cases} (R_n, L_n) \text{ or } (R_n^T, L_n^T), & \text{if } \bullet \overset{M}{\underset{N}{\rightrightarrows}} \bullet \\ (R_n, L_n^T) \text{ or } (R_n, L_n^T), & \text{if } \bullet \overset{M}{\underset{N}{\leftrightarrows}} \bullet \end{cases},$$

where

$$R_n := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 0 & & & 0 \end{bmatrix}, \qquad L_n := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ & & 1 & 0 \\ 0 & & & 0 \end{bmatrix}$$

Main theorem

Each cycle of linear and semilinear mappings is isomorphic to a direct sum, determined uniquely up to isomorphisms of summands, of indecomposable cycles of the following types:

- 

  in which $\mathcal{B}_t$ is given by a Jordan block or an indecomposable canonical block under consimilarity if the number of dashed arrows is even or odd, respectively.
- Cycles that are given by chains.

References

- D. Duarte de Oliveira, V. Futorny, T. Klimchuk, D. Kovalenko, V.V. Sergeichuk, *Cycles of linear and semilinear mappings*, Linear Algebra Appl. 438 (2013) 3442–3453.
- D. Duarte de Oliveira, R.A. Horn, T. Klimchuk, V.V. Sergeichuk, *Remarks on the classification of a pair of commuting semilinear operators*, Linear Algebra Appl. 436 (2012) 3362–3372.

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